FRAMED MODULI SPACES AND TUPLES OF OPERATORS

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Abstract. In this work we address the classical problem of classifying tuples of linear operators and linear functions on a finite dimensional vector space up to base change. Having adopted for the situation considered a construction of framed moduli spaces of quivers, we develop an explicit classification of tuples belonging to a Zariski open subset. For such tuples we provide a finite family of normal forms and a procedure allowing to determine whether two tuples are equivalent.

Introduction

A quiver Q is a diagram of arrows, determined by two finite sets Q_0 (the set of "vertices") and Q_1 (the set of "arrows") with two maps $h, t : Q_1 \to Q_0$ which indicate the vertices at the head and tail of each arrow. A representation W of Q is a collection of (probably infinite dimensional) $\Bbbk$-vector spaces W_i, for each $i \in Q_0$, together with linear maps $W_a : W_{ta} \to W_{ha}$, for each $a \in Q_1$. The dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ of such a representation is given by $\alpha_i = \dim_{\Bbbk} W_i$. A morphism $\psi : W \to U$ of representations consists of linear maps $\psi_i : W_i \to U_i$, for each $i \in Q_0$, such that $\psi_{ha} W_a = U_a \psi_{ta}$, for each $a \in Q_1$. It is an isomorphism if and only if each ψ_i is.

Having chosen vector spaces W_i of dimension α_i, the isomorphism classes of representations of Q with dimension vector α are in natural one-to-one correspondence with the orbits of the group

$$GL(\alpha) := \prod_{i \in Q_0} GL(W_i)$$

in the representation space

$$\mathrm{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \mathrm{Hom}(W_{ta}, W_{ha}).$$

The action is given by $(g \cdot W)_a = g_{ha} W_a g_{ta}^{-1}$, where $g = (g_i)_{i \in Q_0} \in GL(\alpha)$. Note that the one-parameter subgroup $\Delta = \{(tE, \ldots, tE)\}$ acts trivially.

Quivers provide a convenient interpretation of many classical problems of linear algebra. The one we are particularly interested in is classification of tuples of q linear operators and k linear functions on an m-dimensional vector space. In the language of quivers this is equivalent to classification of $(m, 1)$-dimensional representations of $L_{q,k}$, where by $L_{q,k}$ we denote the quiver with two vertices, q loops in the first vertex and k more arrows going from the first vertex to the second one. This problem is known to be wild even for $q = 2$ and $k = 0$, that is no hope remains to write down a complete list of isomorphism classes of representations or even to obtain an algorithm determining whether two given representations are isomorphic. In fact, representation theory of the quiver $L_2 := L_{2,0}$ is proved to be undecidable, see [1] and [6] for a rigorous formulation and a proof of this result.

However, for $\alpha = (m, 1)$ it is possible to explicitly classify representations belonging to a Zariski open subset of $\mathrm{Rep}(L_{q,k}, \alpha)$. The idea comes from the study of stable framed representations.

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Let Q be a quiver and α be a dimension vector. Fix an additional dimension vector ζ and consider the space $\mathrm{Rep}(Q,\alpha,\zeta) := \mathrm{Rep}(Q,\alpha) \oplus \bigoplus_{i\in Q_0} \mathrm{Hom}_{\Bbbk}(\Bbbk^{\alpha_i}, \Bbbk^{\zeta_i})$. Its elements are said to be *framed representations* of Q. Define a $GL(\alpha)$-action on $\mathrm{Rep}(Q,\alpha,\zeta)$ by $g \cdot (M, (f_i)_{i\in Q_0}) = (g \cdot M, (f_i g_i^{-1})_{i\in Q_0})$. A framed representation $(M, (f_i)_{i\in Q_0})$ is called *stable* if there is no nonzero subrepresentation N of M with $M_i \subseteq \ker f_i$, for all $i \in Q_0$. Denote by $\mathrm{Rep}^s(Q,\alpha,\zeta)$ the set of stable framed representations. It is known (see, for example, [8, Theorem 2.3]), that the subset $\mathrm{Rep}^s(Q,\alpha,\zeta)$ admits a geometric quotient, i.e., a morphism to an algebraic variety $\mathcal{M}^s(Q,\alpha,\zeta) := \mathrm{Rep}^s(Q,\alpha,\zeta)/\!\!/GL(\alpha)$ whose fibers coincide with $GL(\alpha)$-orbits. Moreover, for quivers without oriented cycles M. Reineke proved [8, Proposition 3.9] that the quotient space $\mathcal{M}^s(Q,\alpha,\zeta)$ is isomorphic to a Grassmannian of subrepresentations of a certain injective representation of Q. In the general case the quotient is not projective and may not be realized as a Grassmannian of subrepresentations. However, some geometric structure may be revealed by projecting $\mathcal{M}^s(Q,\alpha,\zeta)$ to the categorical quotient $\mathrm{Rep}(Q,\alpha,\zeta)/\!\!/\,\mathrm{GL}(\alpha)$ and studying fibers of this projection, see [2] and [3] for details.

From [7, Proposition 0.9] it follows that the quotient morphism $\mathrm{Rep}^s(Q,\alpha,\zeta) \to \mathcal{M}^s(Q,\alpha,\zeta)$ is a principal fiber bundle. It remains a problem, however, to explicitly describe a finite (and possibly minimal) trivializing covering of the quotient. We construct such a covering using J-skeleta of framed representations, a concept that is a version of the one introduced by K. Bongartz and B. Huisgen-Zimmermann for representations of finite dimensional algebras (see, for example, [4]) adopted and partially simplified to fit our setup. Namely, we show (Theorem 3.3) that $\mathrm{Rep}^s(Q,\alpha,\zeta) = \bigcup_{\mathfrak{S}} X(\mathfrak{S})$, where $X(\mathfrak{S})$ are open subsets parameterized by J-skeleta $\mathfrak{S}$ such that $X(\mathfrak{S}) \cong \mathrm{GL}(\alpha) \times \mathbb{A}^N$, for some positive integer N, and the restriction of the quotient map to $X(\mathfrak{S})$ is the projection onto the second factor.

Framed representations admit another useful interpretation. Consider a new quiver Q^ζ with $Q_0^\zeta = Q_0 \cup \{\infty\}$, the arrow of Q^ζ being those of Q together with ζ_i arrows from i ($i \in Q_0$) to ∞. We also extend the dimension vector α to α^ζ, setting $\alpha_i^\zeta = \alpha_i$ for $i = 1,\ldots,n$ and $\alpha_\infty^\zeta = 1$. It is easy to show that $\mathrm{Rep}(Q,\alpha,\zeta)$ may be identified with $\mathrm{Rep}(Q^\zeta,\alpha^\zeta)$.

Clearly $\mathrm{Rep}(L_{q,k},(m,1))$ is the same as $\mathrm{Rep}(L_q^{(k)}, m^{(k)})$, i.e., as $\mathrm{Rep}(L_q,m,k)$. So, there is a Zariski open subset $\mathrm{Rep}^s(L_{q,k},(m,1))$ of $\mathrm{Rep}(L_{q,k},(m,1))$ where a complete classification of representations is possible. Translating our definition of stable pairs into the language of linear algebra, we may say that $\mathrm{Rep}^s(L_{q,k},(m,1))$ consists of such tuples $(\varphi_1,\ldots,\varphi_q,f_1,\ldots,f_k) \in (\mathrm{End}_{\Bbbk}(\Bbbk^m))^q \oplus ((\Bbbk^m)^*)^k$ that no common proper nonzero invariant subspace of φ_i lies in the common kernel of all f_j. In this paper we show how an explicit classification may be obtained in this setup over an arbitrary field $\Bbbk$.

Section 1 is devoted to exploring a generalized version of the construction introduced in [8]. In Section 2 we define J-skeleta of framed representations and show their existence. In Section 3 we prove that the quotient may be embedded as a locally closed subset in a product of ordinary Grassmannians and construct the above mentioned trivializing covering of $\mathcal{M}^s(Q,\alpha,\zeta)$. Furthermore, we provide a finite family of normal forms for each stable pair and an algorithm allowing to determine whether two stable framed representations are isomorphic. In Section 4 we give a series of examples illustrating how this technique works.

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1. Stable framed representations

Let Q be a quiver with n vertices, α and ζ be two dimension vectors. Choose a vector space $V = \bigoplus_{i\in Q_0} V_i$ with $\dim V_i = \zeta_i$, for $i \in Q_0$. Elements of $\mathrm{Rep}(Q,\alpha,\zeta)$ may be viewed as pairs (M,f), where M is a representation of Q and $f = (f_i : M_i \to V_i)_{i\in Q_0}$ is a map of graded vector spaces.

Recall that a path in Q is a formal product of arrows $a_1 \cdot \ldots \cdot a_k$ such that $t(a_i) = h(a_{i+1})$, for all $i = 1,\ldots,k-1$, or a symbol e_i with $i \in Q_0$. For a path $\tau = a_1 \cdot \ldots \cdot a_k$ we set $t(\tau) = t(a_k)$ and $h(\tau) = h(a_1)$. We also put $h(e_i) = t(e_i) = i$. There is an obvious way to multiply successive paths: if $h(\tau) = t(\sigma)$, the product $\sigma \cdot \tau$ is defined as the concatenation of these paths. All e_i are treated as paths of zero length, that is $e_i^2 = e_i$, for all $i \in Q_0$, and $\tau e_{t(\tau)} = e_{h(\tau)}\tau$, for every path τ.

For each $i \in Q_0$ denote by I_i the following representation of Q. Set

$$(I_i)_j = (\mathrm{span}\,\{\tau \mid \tau : j \rightsquigarrow i \text{ is a path in } Q\})^*,$$

where "$\tau : j \rightsquigarrow i$" means that τ starts in the j-th vertex and ends in the i-th one and $(\cdot)^*$ stands for the dual vector space; in this case $((I_i)_{a:k\to l} f)(\tau) = f(\tau a)$, where $\tau : l \rightsquigarrow i$. This may be rewritten in a more convenient way using the elements in $(I_i)_j$ dual to paths. Namely, to each path $\tau : j \rightsquigarrow i$ in Q we associate an element τ^* in $(I_i)_j$ such that for every $\sigma : j \rightsquigarrow i$ we have

$$\tau^*(\sigma) = \begin{cases} 1, & \text{if } \tau = \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that elements of $(I_i)_j$ may be written as (probably infinite) formal series in τ^*, for $\tau : i \rightsquigarrow j$. In this notation the maps $(I_i)_a$, $a \in Q_1$, are as follows:

$$(I_i)_a(\tau^*) = \begin{cases} \lambda^*, & \text{if } \tau = \lambda a, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the representation $J := \bigoplus_{i\in Q_0} I_i \otimes_{\Bbbk} V_i$. Notice that as a $\Bbbk$-linear space

$$J_i = e_i J \cong \prod_{j\in Q_0} (I_j)_i \otimes_{\Bbbk} V_j \cong \prod_{j\in Q_0} \prod_{\tau: i\rightsquigarrow j} V_j \cong \prod_{\tau:i\rightsquigarrow j} V_j.$$

Sometimes it is convenient to label each component V_j by the corresponding path τ writing $J_i \cong \prod_{\tau:i\rightsquigarrow j} V_j^{(\tau)}$.

Given a point $(M,f) \in \mathrm{Rep}(Q,\alpha,\zeta)$ we define a map $\Phi_{(M,f)} = (\varphi_i)_{i\in Q_0} : M \to J$ by the following rule:

$$\varphi_i = \prod_{\tau:i\rightsquigarrow j} f_j \tau : M_i \to \prod_{\tau:i\rightsquigarrow j} V_j, \tag{1.1}$$

where $\tau(x) := M_{a_1} \ldots M_{a_k}(x)$ for $x \in M_i$ and $\tau = a_1 \ldots a_k$.

The following lemma is straightforward.

Lemma 1.1. *The map $\Phi_{(M,f)}$ is a morphism of representations of Q.*

Proposition 1.2. *The subspace $\ker \Phi_{(M,f)} = \oplus_{i\in Q_0} \ker \varphi_i$ is the maximal $\Bbbk Q$-submodule of M contained in $\ker f$.*

Proof. It follows from Lemma 1.1 that $\ker \Phi_{(M,f)}$ is a $\Bbbk Q$-submodule of M. One also easily observes that $\ker \Phi_{(M,f)} \subseteq \ker f$. Now, let U be a $\Bbbk Q$-submodule of M contained in $\ker f$. For each $\tau : i \rightsquigarrow j$ we then have $\tau U_i = \tau e_i U = \tau U = e_j \tau U \subseteq U_j$. This implies that $f_j(\tau \cdot x) = 0$, for all $x \in U, j \in Q_0$ and for all paths τ, i. e. $U \subseteq \ker \Phi_{(M,f)}$. □

Corollary 1.3. *The map* $\Phi_{(M,f)} : M \to J$ *is injective if and only if the pair* (M, f) *is stable.*

This observation is crucial for the construction. Associated to the maps φ_i are (probably infinite) matrices with rows $f_{iq}\tau$, where $\tau : j \rightsquigarrow i$, $q \in \{1, \dots, \zeta_i\}$, and f_{iq} stands for the q-th row of the matrix of f_i. The map is injective if and only if one of $\alpha_i \times \alpha_i$ minors of its matrix is nonzero, i.e., for some $\tau_1, \dots, \tau_{\alpha_i}$ and $q_1, \dots, q_{\alpha_i}$, we have

$$\det \begin{pmatrix} f_{iq_1}\tau_1 \\ f_{iq_2}\tau_2 \\ \vdots \\ f_{iq_{\alpha_i}}\tau_{\alpha_i} \end{pmatrix} \neq 0.$$

Therefore, a pair (M, f) is stable if and only if for each $i \in Q_0$ there is a set of numbers $q_1^{(i)}, \dots, q_{\alpha_i}^{(i)}$ and a set of distinct paths $\tau_1^{(i)}, \dots, \tau_{\alpha_i}^{(i)}$ with

$$D^{(q_1^{(1)},\dots,q_{\alpha_1}^{(1)};\dots;q_1^{(n)},\dots,q_{\alpha_n}^{(n)})}_{(\tau_1^{(1)},\dots,\tau_{\alpha_1}^{(1)};\dots;\tau_1^{(n)},\dots,\tau_{\alpha_n}^{(n)})} := \det \begin{pmatrix} f_{iq_1^{(1)}}\tau_1^{(1)} \\ f_{iq_2^{(1)}}\tau_2^{(1)} \\ \vdots \\ f_{iq_{\alpha_1}^{(1)}}\tau_{\alpha_1}^{(1)} \end{pmatrix} \cdot \ldots \cdot \det \begin{pmatrix} f_{iq_1^{(n)}}\tau_1^{(n)} \\ f_{iq_2^{(n)}}\tau_2^{(n)} \\ \vdots \\ f_{iq_{\alpha_n}^{(n)}}\tau_{\alpha_n}^{(n)} \end{pmatrix} \neq 0.$$

Thus $\mathrm{Rep}^s(Q, \alpha, \zeta)$ is covered by open subsets $U^{(q_1^{(1)},\dots,q_{\alpha_1}^{(1)};\dots;q_1^{(n)},\dots,q_{\alpha_n}^{(n)})}_{(\tau_1^{(1)},\dots,\tau_{\alpha_1}^{(1)};\dots;\tau_1^{(n)},\dots,\tau_{\alpha_n}^{(n)})}$, which are the nonzero loci of the corresponding $D^{(q_1^{(1)},\dots,q_{\alpha_1}^{(1)};\dots;q_1^{(n)},\dots,q_{\alpha_n}^{(n)})}_{(\tau_1^{(1)},\dots,\tau_{\alpha_1}^{(1)};\dots;\tau_1^{(n)},\dots,\tau_{\alpha_n}^{(n)})}$, and consequently the moduli space is covered by the quotients $V^{(q_1^{(1)},\dots,q_{\alpha_1}^{(1)};\dots;q_1^{(n)},\dots,q_{\alpha_n}^{(n)})}_{(\tau_1^{(1)},\dots,\tau_{\alpha_1}^{(1)};\dots;\tau_1^{(n)},\dots,\tau_{\alpha_n}^{(n)})} := U^{(q_1^{(1)},\dots,q_{\alpha_1}^{(1)};\dots;q_1^{(n)},\dots,q_{\alpha_n}^{(n)})}_{(\tau_1^{(1)},\dots,\tau_{\alpha_1}^{(1)};\dots;\tau_1^{(n)},\dots,\tau_{\alpha_n}^{(n)})} /\!\!/ GL(\alpha)$. In next section we shall prove that this covering admits a finite subcovering.

2. Skeleta of stable pairs

Let Q, α and ζ be as before. Consider a quiver Q^ζ with $Q_0^\zeta = Q_0 \cup \{\infty\}$, the arrows of Q^ζ being those of Q together with ζ_i arrows from each $i \in Q_0$ to ∞. Denote the new arrows by f_{iq}, where i indicates the tail of an arrow and $q \in \{1, \dots, \zeta_i\}$. We also extend the dimension vector α to α^ζ, setting $\alpha_i^\zeta = \alpha_i$ for $i = 1, \dots, n$ and $\alpha_\infty^\zeta = 1$.

Observe that the sets $\mathrm{Rep}(Q, \alpha, \zeta)$ and $\mathrm{Rep}(Q^\zeta, \alpha^\zeta)$ may be identified in a $\mathrm{GL}(\alpha)$-invariant way. Indeed,

$$\mathrm{Rep}(Q, \alpha, \zeta) = \mathrm{Rep}(Q, \alpha) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}_{\Bbbk}(\Bbbk^{\alpha_i}, \Bbbk^{\zeta_i}) \cong$$

$$\cong \mathrm{Rep}(Q, \alpha) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}_{\Bbbk}(\Bbbk^{\alpha_i}, \Bbbk)^{\zeta_i} = \mathrm{Rep}(Q^\zeta, \alpha^\zeta).$$

In terms of matrices this isomorphism has the following interpretation. Let (M, f) be a framed representation and $\widetilde{M}$ be the corresponding representation of Q^ζ. Then matrices of $\widetilde{M}_{f_{iq}}$, $q = 1, \dots, \zeta_i$ are rows of the matrix of f_i (i.e., what was denoted by f_{iq} in Section 1). This justifies our seeming abuse of notation.

Example 2.1. Let $Q = \begin{tikzcd} 1 \arrow[r, "b"] \arrow[rr, bend right, "a"'] & 2 \arrow[r, "c"] & 3 \end{tikzcd}$ and $\zeta = (1,2,3)$. Then we have

$$Q^\zeta = \begin{tikzcd} & \infty & \\ 1 \arrow[ur, bend left, "f_{11}"] \arrow[r, "b"] \arrow[rr, bend right, "a"'] & 2 \arrow[u, bend left, "f_{21}"] \arrow[u, bend right, "f_{22}"'] \arrow[r, "c"] & 3 \arrow[ul, "f_{31}"'] \arrow[ul, bend right, "f_{32}"'] \arrow[ul, bend right=40, "f_{33}"'] \end{tikzcd}$$

Furthermore, if $\alpha = (2,2,1)$ and

$$(M,f) = \begin{tikzcd} \Bbbk & \Bbbk^2 & \Bbbk^3 \\ \Bbbk^2 \arrow[u, "(1\ 5)"] \arrow[r, "M_b"] \arrow[rr, bend right, "M_a"'] & \Bbbk^2 \arrow[u, "\left(\begin{smallmatrix} 3 & 1 \\ 4 & 2 \end{smallmatrix}\right)"] \arrow[r, "M_c"] & \Bbbk \arrow[u, "\left(\begin{smallmatrix} 3 \\ 0 \\ 1 \end{smallmatrix}\right)"] \end{tikzcd}$$

for some M_a, M_b and M_c, then the corresponding representation $\widetilde{M}$ of Q^ζ is as follows

$$\widetilde{M} = \begin{tikzcd} & \Bbbk & \\ \Bbbk^2 \arrow[ur, bend left, "(1\ 5)"] \arrow[r, "M_b"] \arrow[rr, bend right, "M_a"'] & \Bbbk^2 \arrow[u, bend left, "(3\ 1)"] \arrow[u, bend right, "(4\ 2)"'] \arrow[r, "M_c"] & \Bbbk \arrow[ul, "(3)"'] \arrow[ul, bend right, "(0)"'] \arrow[ul, bend right=40, "(1)"'] \end{tikzcd}$$

The representation J may also be extended in a natural way to a representation $\widetilde{J}$ of Q^ζ. Set $\widetilde{J}_i = J_i$ and $\widetilde{J}_a = J_a$, for $i \in Q_0$ and $a \in Q_1$. Set further $\widetilde{J}_\infty = \Bbbk$ and $\bigoplus_{b:i\to\infty} J_b : J_i \to \Bbbk^{\zeta_i}$ be the projection $\prod_{\tau:i\rightsquigarrow j} V_j^{(\tau)} \to V_i^{(e_i)}$, for each $i \in Q_0$. A straightforward calculation shows that thus constructed $\widetilde{J}$ is isomorphic to the representation I_∞ of Q^ζ. In particular, elements of $\widetilde{J}_i$ may be represented as (possibly infinite) formal series in $(f_{jq}\tau)^*$, for $j \in Q_0$, $q = 1,\dots,\zeta_j$, and $\tau : i \rightsquigarrow j$. This implies that paths in Q^ζ may be viewed as linear functions on $\widetilde{J}$.

For a framed representation $(M,f) \in \operatorname{Rep}(Q,\alpha,\zeta)$ consider the corresponding representation $\widetilde{M}$ of Q^ζ. The map $\Phi_{(M,f)}$ induces then a morphism $\widetilde{\Phi}_{\widetilde{M}} : \widetilde{M} \to \widetilde{J}$ of representations of Q^ζ defined by $\widetilde{\varphi}_i = \varphi_i$, for $i \in Q_0$, $\widetilde{\varphi}_\infty = \mathrm{id}_\Bbbk$. It follows from Corollary 1 that a pair (M,f) is stable if and only if the corresponding map $\widetilde{\Phi}_{\widetilde{M}}$ is an embedding.

Definition. By a *J-skeleton of a stable pair (M,f)* we understand a set $\mathfrak{S}$ of paths of nonzero length in Q^ζ ending in ∞ with the following properties:

(1) Restrictions of paths in $\mathfrak{S}$ together with e_∞ give a basis in $\widetilde{\Phi}_{\widetilde{M}}\left(\widetilde{M}\right)^*$.
(2) Whenever τa is in $\mathfrak{S}$ and $\tau \neq e_\infty$, τ is also in $\mathfrak{S}$.

The *dimension vector* of a J-skeleton $\mathfrak{S}$ is the dimension vector $\alpha = \underline{\dim}(\mathfrak{S})$ of any stable pair with J-skeleton $\mathfrak{S}$. It is easy to see that α_i equals the number of paths in $\mathfrak{S}$ ending on f_{iq} for any $q = 1,\dots,\zeta_i$. For $i \in Q_0$ we set $\mathfrak{S}_i = \{f_{iq}\tau \in \mathfrak{S}\}$.

Lemma 2.2. *Every stable pair (M,f) has a J-skeleton.*

Proof. Let (M,f) be a stable pair. Let also $\widetilde{M}$ be the corresponding representation of Q^ζ. Denote by N its image $\widetilde{\Phi}_{\widetilde{M}}(\widetilde{M}) \subseteq \widetilde{J}$. First of all, we need to show that restrictions of paths

in Q generate $\widetilde{\Phi}_{\widetilde{M}}(\widetilde{M})^*$ as a vector space. Let $\varpi_1, \dots, \varpi_m$ be a basis of $\widetilde{\Phi}_{\widetilde{M}}(\widetilde{M})$, where $\varpi_i = \sum c_{iq,\tau}(f_{iq}\tau)^*$. Observe that, for some t,

$$\dim\left(\operatorname{span}\left\{\sum_{f_{i_q}\tau,\ l(\tau)\leqslant t} c_{i_q,\tau}(f_{iq}\tau)^*\right\}\right) = m,$$

where $l(\tau)$ stands for the length of a path τ. Then, obviously, the linear span of all paths of length not greater than t generates $\widetilde{\Phi}_{\widetilde{M}}(\widetilde{M})^*$.

We see now, that restrictions of paths in Q give a basis of $\Phi_{\widetilde{M}}(\widetilde{M})^*$, but less evident is condition (2).

We construct its J-skeleton inductively. We start by taking, for each $i \in Q_0$, a maximal tuple $f_{iq_1}, \dots, f_{iq_t}$ with $f_{iq_1}|_N, \dots, f_{iq_t}|_N$ linearly independent. Further, on each step we add a path τa, where τ is one of the paths we added before, a is an arrow, and $\tau a|_N$ does not lie in the linear span of restrictions of all the preceding paths. We proceed until we obtain a maximal linearly independent set of restrictions Γ. It should be proved, however, that it is a basis of N^*. Let $\tau|_N \notin \operatorname{span}\{\Gamma\}$. If none of its final subpaths restricted to N is in $\operatorname{span}\{\Gamma\}$, then we have found f_{lq_l} whose restriction is not in $\operatorname{span}\{\Gamma\}$, a contradiction. Thus $\tau = \mu\nu$, where $\mu|_N \in \operatorname{span}\{\Gamma\}$, i.e., $\mu|_N = \sum_{\kappa\in\Gamma} c_\kappa \kappa|_N$. Consequently, $\tau|_N = \sum_{\kappa\in\Gamma} c_\kappa \kappa|_N \nu|_N$. But by maximality of Γ each $\kappa|_N \nu|_N$ is in $\operatorname{span}\{\Gamma\}$. □

Obviously, $\operatorname{Rep}^s(Q,\alpha,\zeta) = \bigcup_{\mathfrak{S}} \operatorname{Rep}(Q,\mathfrak{S})$, where

$$\operatorname{Rep}(Q,\mathfrak{S}) = \left\{(M,f) \in \operatorname{Rep}^s(Q^\zeta,\alpha,\zeta) \mid (M,f) \text{ has } J\text{-skeleton } \mathfrak{S}\right\}$$

and $\mathfrak{S}$ run through all possible J-skeleta for dimension vectors α and ζ. Now note that if $\mathfrak{S} = \left\{f_{1q_1^{(1)}}\tau_1^{(1)}, \dots, f^{(1)}_{q_{1\alpha_1}}\tau^{(1)}_{\alpha_1}, \dots, f_{nq_1^{(n)}}\tau_1^{(n)}, \dots, f^{(n)}_{q_{n\alpha_n}}\tau^{(n)}_{\alpha_n}\right\}$, then $\operatorname{Rep}(Q,\mathfrak{S})$ is exactly the open subset $U^{(q_1^{(1)},\dots,q_{\alpha_1}^{(1)};\dots;q_1^{(n)},\dots,q_{\alpha_n}^{(n)})}_{(\tau_1^{(1)},\dots,\tau_{\alpha_1}^{(1)};\dots;\tau_1^{(n)},\dots,\tau_{\alpha_n}^{(n)})}$. Observe that there are only finitely many J-skeleta, since lengths of paths in an α-dimensional J-skeleton are bounded by $\max_i \alpha_i$. Therefore, we obtain a finite covering of $\operatorname{Rep}^s(Q,\alpha,\zeta)$ by subsets of form $U^{(q_1^{(1)},\dots,q_{\alpha_1}^{(1)};\dots;q_1^{(n)},\dots,q_{\alpha_n}^{(n)})}_{(\tau_1^{(1)},\dots,\tau_{\alpha_1}^{(1)};\dots;\tau_1^{(n)},\dots,\tau_{\alpha_n}^{(n)})}$.

3. Embedding of the moduli space

Let $\Gamma(\alpha)$ be the set of all paths occurring in J-skeleta with dimension vector α and $\widetilde{\Gamma}(\alpha)$ be the union of Γ with $\{\tau a \mid \tau \in \Gamma(\alpha), a \in Q_1, h(a) = t(\tau)\}$. Let also $\widehat{J} = \bigoplus_{\tau\in\widetilde{\Gamma}(\alpha)} V^{(\tau)}_{h(\tau)}$. Note that $\widehat{J}$ has a natural Q_0-grading. Indeed, one may set $\widehat{J}_i = \bigoplus_{\tau\in\widetilde{\Gamma}(\alpha),t(\tau)=i} V^{(\tau)}_{h(\tau)}$. Consider the map $\widehat{\Phi}_{(M,f)}$: $M \to \widehat{J}$ defined as follows:

$$\widehat{\varphi}_i = \bigoplus_{\tau\in\widetilde{\Gamma}(\alpha)} f_{h(\tau)}\tau : M_i \to \bigoplus_{\tau\in\widetilde{\Gamma}(\alpha),t(\tau)=i} V^{(\tau)}_{h(\tau)} = \widehat{J}_i.$$

It is obvious that a pair (M,f) is stable if and only if $\widehat{\Phi}_{(M,f)}$ is injective. The advantage of $\widehat{\Phi}_{(M,f)}$ is that it maps to a finite dimensional vector space.

We now need to fix notation that will be used throughout the rest of the paper. Let $\widetilde{\Gamma}_i(\alpha)$ be a subset of $\widetilde{\Gamma}(\alpha)$ consisting of paths starting at i. Let further $(B^{(1)}, \dots, B^{(n)}) \in \prod_{i=1}^n \operatorname{Mat}_{|\widetilde{\Gamma}_i(\alpha)|\times\alpha_i}(\Bbbk)$. By definition of $\widetilde{\Gamma}(\alpha)$ rows of $B^{(i)}$ correspond to paths in Q^ζ starting at i. So, instead of using numerical indices we denote, for a path τ in Q, the row of $B^{(t(\tau))}$ corresponding to τ by B_τ. Let, furthermore, $\mathfrak{T}$ be a subset of $\widetilde{\Gamma}(\alpha)$ (not necessary a skeleton). Denote by $\mathfrak{T}_i$ the subset in $\mathfrak{T}$ consisting of paths starting

at i. Set $\underline{\dim}(\mathfrak{T}) = (|\mathfrak{T}_1|, \ldots, |\mathfrak{T}_n|)$. Finally, for a collection $\mathfrak{T}$ with dimensional vector α let $B(\mathfrak{T}_i)$ be the submatrix of $B^{(i)}$ composed of rows corresponding to paths in $\mathfrak{T}_i$ and $U(\mathfrak{T})$ be the open subset in $\prod_{i=1}^n \mathrm{Mat}_{|\widetilde{\Gamma}_i(\alpha)|\times\alpha_i}(\Bbbk)$ consisting of tuples $(B^{(1)}, \ldots, B^{(n)})$ with all $B(\mathfrak{T}_i)$ nondegenerate.

For a Q_0-graded space $W = \bigoplus_{i\in Q_0} W_i$ define $\mathrm{IHom}_\alpha(W) := \prod_{i\in Q_0} \mathrm{IHom}_{\alpha_i}(W_i)$, where $\mathrm{IHom}_{\alpha_i}(W_i)$ is the set of all injective linear maps from the vector space $\Bbbk^{\alpha_i}$ to W_i. We also define $\mathrm{Gr}_\alpha(W)$ as the product of Grassmannians $\prod_{i\in Q_0} \mathrm{Gr}_{\alpha_i}(W_i)$. It is easy to see that $\mathrm{Gr}_\alpha(W)$ is a quotient of $\mathrm{IHom}_\alpha(W)$ by the natural action of $GL(\alpha)$. We now introduce a map

$$\widehat{\Phi} : \mathrm{Rep}^s(Q, \alpha, \zeta) \to \mathrm{IHom}_\alpha(\widehat{J}), \quad (M, f) \mapsto \widehat{\Phi}_{(M,f)}.$$

Identify $\mathrm{IHom}_\alpha(\widehat{J})$ with an open subset

$$\bigcup_{\mathfrak{T}:\, \underline{\dim}\mathfrak{T}=\alpha} U(\mathfrak{T}) \subseteq \prod_{i=1}^n \mathrm{Mat}_{|\widetilde{\Gamma}_i(\alpha)|\times\alpha_i}(\Bbbk).$$

It is easy to see that $\operatorname{Im}\widehat{\Phi}$ is contained in

$$Z(\alpha) := \bigcup_{\substack{\mathfrak{S} \text{ is a } J\text{-skeleton} \\ \underline{\dim}(\mathfrak{S})=\alpha}} U(\mathfrak{S}).$$

Definition. For a J-skeleton $\mathfrak{S}$ set $X(\mathfrak{S}) = \operatorname{Im}(\widehat{\Phi}) \cap U(\mathfrak{S})$.

Proposition 3.1. *The image of $\widehat{\Phi}$ is a locally closed subvariety in $\mathrm{IHom}_\alpha(\widehat{J})$.*

Proof. It is sufficient to show that each $X(\mathfrak{S})$ is closed in $U(\mathfrak{S})$. Fix a J-skeleton $\mathfrak{S}$. For an arrow $a \in Q_1$ set $\mathfrak{S}a = \{\tau a \mid \tau \in \mathfrak{S},\, h(a) = t(\tau)\}$. If a tuple of matrices $(B^{(1)}, \ldots, B^{(n)}) \in U(\mathfrak{S})$ is in $\operatorname{Im}\widehat{\Phi}$, we can recover all the maps M_a, $a \in Q_1$ of its inverse image (M, f) since for all $a \in Q_1$: $B(\mathfrak{S}_{ta}a) = B(\mathfrak{S}_{ta})M_a$ and hence $M_a = B(\mathfrak{S}_{ta})^{-1}B(\mathfrak{S}_{ta}a)$. Now observe that $(B^{(1)}, \ldots, B^{(n)})$ belongs to the image of $\widehat{\Phi}$ whenever, for all $\tau \in \Gamma(\alpha)$ and a with $ta = h(\tau)$, $B_{\tau a} = B_\tau M_a$. Using the expression received for a, we rewrite this as

$$B_{\tau a} = B_\tau B(\mathfrak{S}_{ta})^{-1} B(\mathfrak{S}_{ta}a). \tag{3.1}$$

Collected together, all these equations define a Zarisky closed subvariety $X(\mathfrak{S})$ in $U(\mathfrak{S})$. Gluing them we obtain a closed subvariety $X_0(\alpha) \subseteq Z(\alpha)$ that coincides with $\operatorname{Im}\widehat{\Phi}$. Consequently, $\operatorname{Im}\widehat{\Phi}$ is a locally closed subvariety in $\mathrm{IHom}_\alpha\left(\widehat{J}\right)$. $\square$

From the proof of this proposition we infer

Corollary 3.2. *The map $\widehat{\Phi} : \mathrm{Rep}^s(Q, \alpha, \zeta) \to \operatorname{Im}\widehat{\Phi} = X_0(\alpha)$ is a $GL(\alpha)$-equivariant isomorphism of algebraic varieties.*

Proof. To construct the inverse morphism we need to find ways of recovering a pair (M, f) possessing its image $(B^{(1)}, \ldots, B^{(n)}) \in \mathrm{IHom}_\alpha(\widehat{J})$. But f_{iq} comes as $B_{f_{iq}}$ and $M_a = B(\mathfrak{S}_{ta})^{-1}B(\mathfrak{S}_{ta}a)$ in $U(\mathfrak{S})$ while on intersections $U(\mathfrak{S})\cap U(\mathfrak{T})$ the equality $B(\mathfrak{S}_{ta})^{-1}B(\mathfrak{S}_{ta}a) = B(\mathfrak{T}_{ta})^{-1}B(\mathfrak{T}_{ta}a)$ is a direct consequence of the equations defining X_0.

The equivariance of this isomorphism is obvious. $\square$

From now on we will identify each $X(\mathfrak{S}) \subseteq \mathrm{IHom}_\alpha(\widehat{J})$ with its preimage $\mathrm{Rep}(Q, \mathfrak{S}) = \widehat{\Phi}^{-1}(X(\mathfrak{S})) \subseteq \mathrm{Rep}^s(q, \alpha, \zeta)$ and treat it as a subset in $\mathrm{Rep}^s(Q, \alpha, \zeta)$ whenever needed.

Theorem 3.3. *In the notation given above*

(1) *We have*

$$\mathrm{Rep}^s(Q,\alpha,\zeta) = \bigcup_{\mathfrak{S} \text{ is a } J\text{-skeleton}} X(\mathfrak{S}),$$

where $X(\mathfrak{S})$ are Zariski open subsets of $\mathrm{Rep}^s(Q,\alpha,\zeta)$ such that $X(\mathfrak{S}) \cong \mathrm{GL}(\alpha) \times \mathbb{A}^N$, for some positive integer N, and the restriction to $X(\mathfrak{S})$ of the quotient map is the projection onto the second factor. In particular,

$$\mathcal{M}^s(Q,\alpha,\zeta) = \bigcup_{\mathfrak{S} \text{ is a } J\text{-skeleton}} X(\mathfrak{S})/\!\!/GL(\alpha)$$

is a covering by open subspaces isomorphic to affine spaces.

(2) *The quotient space $\mathcal{M}^s(Q,\alpha,\zeta)$ is isomorphic to a locally closed subvariety in $\mathrm{Gr}_\alpha(\widehat{J})$.*

Proof. Again it will be convenient for us to view $\mathrm{IHom}_\alpha(\widehat{J})$ as a Zariski open subset in $\prod_{i\in Q_0} \mathrm{Mat}_{|\widetilde{\Gamma}_i(\alpha)|\times\alpha_i}(\Bbbk)$. Fix a J-skeleton $\mathfrak{S}$ and assume that its elements are somehow ordered (for example, lexicographically). For a tuple of matrices $B = (B^{(1)},\dots,B^{(n)})$ define $B_{[\mathfrak{S}]}$ as the n-tuple of matrices with $B^{(i)}_{[\mathfrak{S}]}$ a submatrix of $B^{(i)}$ consisting of all the rows of all $B(\mathfrak{S}a)$, for $a \in Q_1$ with $t(a) = i$, that do not occur in $B(\mathfrak{S}_i)$. Also denote by $B_{\widehat{\mathfrak{S}}}$ a tuple of matrices with $B^{(i)}_{\widehat{\mathfrak{S}}}$ obtained as a union of $B^{(i)}_{[\mathfrak{S}]}$ with all $B_{f_{ik_i}}$, for $f_{ik_i} \notin \mathfrak{S}$. Denote by T_i the number of rows in $B^{(i)}_{[\mathfrak{S}]}$.

Consider the morphism $\pi_{\mathfrak{S}} : \prod_{i\in Q_0} \mathrm{Mat}_{|\widetilde{\Gamma}_i(\alpha)|\times\alpha_i}(\Bbbk) \to \prod_{i\in Q_0} \mathrm{Mat}_{T_i\times\alpha_i}(\Bbbk)$ defined by $B^{(i)} \mapsto \big(B^{(i)} \cdot B(\mathfrak{S}_i)^{-1}\big)_{\widehat{\mathfrak{S}}}$. We claim that it provides the quotient morphism for the action of $GL(\alpha)$ on $U(\mathfrak{S})$.

First of all, $\pi_{\mathfrak{S}}$ is $GL(\alpha)$-invariant, since, for $g \in GL(\alpha)$, we have $\pi_{\mathfrak{S}}(g\cdot B)^{(i)} = \pi_{\mathfrak{S}}(Bg^{-1})^{(i)} = \big(B^{(i)}g_i^{-1} \cdot g_i B(\mathfrak{S}_i)^{-1}\big)_{\widehat{\mathfrak{S}}} = \pi_{\mathfrak{S}}(B_i)$. We now prove that $\pi_{\mathfrak{S}}$ is surjective. Let $C \in \prod_{i\in Q_0} \mathrm{Mat}_{T_i\times\alpha_i}(\Bbbk)$. Recall that each row of each $C^{(i)}$ corresponds to a path from $\bigcup_{a\in Q_1}(\mathfrak{S}a\backslash\mathfrak{S}) \cup \{f_i \notin \mathfrak{S}\}$. Take an identity $\alpha_i \times \alpha_i$-matrix $E^{(i)}$ and put its j-th row into correspondence with the j-th path from $\mathfrak{S}_i$ (with respect to the order we introduced at the beginning of the proof). Now, add to $C^{(i)}$ all the rows of $E^{(i)}$ corresponding to paths from $\mathfrak{S}\cap(\bigcup_{a\in Q_1,\, ta=i}\mathfrak{S}a)$ and denote the matrix received by $\widetilde{C}^{(i)}$. The first step will be now to recover a stable pair (M^C, f^C), then we will use it to obtain a matrix in $\pi_{\mathfrak{S}}^{-1}(C)$. Put $M^C_a = \widetilde{C}(\mathfrak{S}a)$, for all $a \in Q_1$, and

$$f_i^C = \begin{cases} E^{(i)}_{f_i}, & \text{if } f_i \in \mathfrak{S}, \\ C^{(i)}_{f_i}, & \text{otherwise.} \end{cases}$$

Finally, set $B^C = \Phi_{(M^C,f^C)}$. One should show now that $\pi_{\mathfrak{S}}(B^C) = C$. But $B(\mathfrak{S}_i) = E^{(i)}$, so $B(\mathfrak{S}a) = B(\mathfrak{S})a = M_a = \widetilde{C}(\mathfrak{S}a)$, for all $a \in Q_1$, implying that $\pi_{\mathfrak{S}}(B)_{[\mathfrak{S}]} = B_{[\mathfrak{S}]} = C_{[\mathfrak{S}]}$, for $a \in Q_1$. Analogously, $B^{(i)}_{f_i} = C^{(i)}_{f_i}$, for all $f_i \notin \mathfrak{S}$. Thus, $\pi_{\mathfrak{S}}(B) = C$ and the surjectivity is proven.

Now we should show that fibers of $\pi_{\mathfrak{S}}$ coincide with $GL(\alpha)$-orbits in $U(\mathfrak{S})$. Observe, that for the above constructed $B^C \in \pi_{\mathfrak{S}}^{-1}(C)$ we have $\big((B^C)^{(i)} \cdot B^C(\mathfrak{S}_i)^{-1}\big)(\mathfrak{S}) = E^{(i)}$, an identity matrix. But it is easy to see that a $GL(\alpha)$-orbit in $U(\mathfrak{S})$ contains only one tuple of matrices with this property. So, any $B \in \pi_{\mathfrak{S}}^{-1}(C)$ equals $g(B)\cdot B^C$, where $g(B)_i = B(\mathfrak{S}_i)$.

An isomorphism $\mathrm{GL}(\alpha)\times\prod_{i\in Q_0}\mathrm{Mat}_{T_i\times\alpha_i}(\Bbbk)\to X(\mathfrak{S})$ is therefore obtained by sending a pair (g,C) to $g\cdot B^C=\left((B^C)^{(i)}g_i^{-1}\right)_{i\in Q_0}$.

To prove the second part it is sufficient to check that each $X(\mathfrak{S})/\!\!/GL(\alpha)$ embeds into $\mathrm{Gr}(\mathfrak{S}):=U(\mathfrak{S})/\!\!/GL(\alpha)$ as a locally closed subvariety. Observe that the natural projection $\pi_0: U(\mathfrak{S})\twoheadrightarrow\mathrm{Gr}(\mathfrak{S})$ may be viewed as the map $\prod_{i\in Q_0}\mathrm{Mat}_{|\widetilde{\Gamma}_i(\alpha)|\times\alpha_i}(\Bbbk)\to\prod_{i\in Q_0}\mathrm{Mat}_{(|\widetilde{\Gamma}_i(\alpha)|-\alpha_i)\times\alpha_i}(\Bbbk)$, $B\mapsto\left(B^{(i)}\cdot B(\mathfrak{S}_i)^{-1}\right)_{\Gamma(\alpha)\setminus\mathfrak{S}}$. Hence $X(\mathfrak{S})/\!\!/GL(\alpha)$ is isomorphic to a subvariety in $\mathrm{Gr}(\mathfrak{S})$ defined by equations

$$C_{\tau a}=C_\tau C(\mathfrak{S}_{ta}a),$$

for all paths τ and arrows a with $\tau a\notin\left(\bigcup_{a\in Q_1,\,ta=i}\mathfrak{S}a\right)$. □

Corollary 3.4. *In the settings of Theorem 1 we have the following*

1. *Each stable pair (M,f) possesses a finite family of normal forms, each normal form corresponding to a J-skeleton of (M,f). If $\mathfrak{S}$ is a J-skeleton of (M,f), then the respective normal form of (M,f) is $\left(M^C,f^C\right)$, where $C=\pi_{\mathfrak{S}}(M,f)$.*
2. *The following procedure may be used to determine whether two stable pair (M,f) and (M',f') are isomorphic.*
   (a) *Compute tuples of matrices B and B' corresponding to (M,f) and (M',f').*
   (b) *Find their J-skeleta by seeking non-degenerate maximal minors of B and B'.*
   (c) *If they have no common J-skeleta, the pairs are not isomorphic.*
   (d) *If $\mathfrak{S}$ a common J-skeleton, compute $\left(M^C,f^C\right)$ and $\left(M^{C'},f^{C'}\right)$ for $C=\pi_{\mathfrak{S}}(B)$ and $C'=\pi_{\mathfrak{S}}(B')$.*
   (e) *The pairs (M,f) and (M',f') are isomorphic if and only if $\left(M^C,f^C\right)=\left(M^{C'},f^{C'}\right)$.*

Remark 3.5. Dimensions of the affine spaces covering the quotient space equal the dimension of the quotient itself, i.e., the difference $\dim\mathrm{Rep}(Q,\alpha,\zeta)-\dim\mathrm{GL}(\alpha)$. For $Q=L_{q,k}$ we have $\dim X(\mathfrak{S})/\!\!/\mathrm{GL}_m=\dim\mathrm{Rep}(L_q,m,k)-\dim\mathrm{GL}_m=(m^2q+mk)-m^2=m(mq+k-m)$, so that $X(\mathfrak{S})/\!\!/GL_m\cong\mathbb{A}^{m(mq+k-m)}$.

We have described affine charts covering the quotient. Since all of them are affine spaces, on each we obtain a convenient system of local coordinates. The transition functions between these coordinates may be described in the following way. Let $\mathfrak{S}$ and $\mathfrak{T}$ be two J-skeleta, and $B\in\prod_{i\in Q_0}\mathrm{Mat}_{|\widetilde{\Gamma}_i(\alpha)|\times\alpha_i}(\Bbbk)$ be a matrix representing a point in $\mathrm{Rep}^s(Q,\alpha,\zeta)$. Then we may express B in matrix elements of $(B^{(i)}\cdot B(\mathfrak{S}_i)^{-1})_{\widehat{\mathfrak{S}}}$, for $i\in Q_0$, and further obtain the expression for $(B^{(j)}\cdot B(\mathfrak{T}_j)^{-1})_{\widehat{\mathfrak{T}}}$.

The same procedure may be used to establish relations between normal forms of a stable pair constructed using different J-skeleta.

4. Examples

Let Q be the quiver L_q and $\alpha=(m)$. It is easy to see that every J-skeleton that may occur in this case is a subset of $\{f_iW(a_1,\dots,a_q)\mid W$ is a word in a_j of length at most $m-1\}$, so we set

$$\widetilde{\Gamma}(m)=\{f_iW(a_1,\dots,a_q)\mid W\text{ is a word in }a_j\text{ of length at most }m\},$$

$\widehat{J}=\bigoplus_{i,W,length(W)\leqslant m}\Bbbk u^{i,W}$ and $\varphi_{i,W}(m)=f_iW(a_1,\dots,a_q)m$.

Example 4.1. Let $q = k = 1$. This encodes the problem of classifying pairs (a, f), where a is a linear operator on $\Bbbk^m$ and f is a linear function. The extended quiver $L_1^{(1)} = L_{1,1}$ is

$$\begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, "f"] & \infty \end{tikzcd}$$

There is only one J-skeleton, namely $\mathfrak{S} = \{f, fa, \ldots, fa^{m-1}\}$. Therefore, $\widetilde{\Gamma}(m) = \{f, fa, \ldots, fa^m\}$ and $\widehat{J} = \bigoplus_{i=0}^m V_1^{(fa^i)}$, where $V_1 = \Bbbk$, so that $\widehat{J} = \Bbbk^{m+1}$ as a vector space. Furthermore, we have

$$\widehat{\Phi} : \mathrm{Rep}(L_1, m, 1) \to \mathrm{Mat}_{(m+1)\times m}(\Bbbk), \quad (a, f) \mapsto \widehat{\Phi}_{(a,f)} = \begin{pmatrix} f \\ fa \\ \vdots \\ fa^m \end{pmatrix}.$$

One can easily see that the stability condition (that is injectivity of $\widehat{\Phi}_{(a,f)}$) is in this case equivalent to f being a cyclic vector for the natural action $GL_m : (\Bbbk^m)^*$. The subset $X(\mathfrak{S}) \subseteq \mathrm{Mat}_{(m+1)\times m}(\Bbbk)$ coincides with $U(\mathfrak{S})$, i.e., consists of matrices $B = (b_{ij})$ with first m rows linear independent. Indeed, the only condition $B_{fa^m} = B_{fa^{m-1}} B(\mathfrak{S})^{-1} B(\mathfrak{S}a)$ holds identically. Next, for a $(m+1) \times m$-matrix A we have $A_{\widehat{\mathfrak{S}}} = A_{fa^m}$, hence the quotient map $\pi_{\mathfrak{S}}$ sends

$$\pi_{\mathfrak{S}} : B = (b_{ij}) \mapsto (b_{m+1,1}, \ldots, b_{m+1,m}) \begin{pmatrix} b_{11} & \ldots & b_{1m} \\ \vdots & & \vdots \\ b_{m1} & \ldots & b_{mm} \end{pmatrix}^{-1}.$$

The quotient is isomorphic to $\mathbb{A}^m$ and $\pi_{\mathfrak{S}}$ admits a section $\mathbb{A}^m \to \mathrm{Rep}(L_1, m, 1)$ sending $C = (c_0, \ldots, c_{m-1})$ to the pair

$$f^C = (1, 0, \ldots, 0), \qquad a^C = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & 1 & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ c_0 & c_1 & \ldots & c_{m-2} & c_{m-1} \end{pmatrix},$$

where c_i are the coefficients of the characteristic polynomial of a.

Example 4.2. Let $q = 1$, $m = k = 2$. The corresponding extended quiver $L_1^{(2)} = L_{1,2}$ is

$$\begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & \infty \end{tikzcd}$$

There are three possible J-skeleta: $\mathfrak{S}_1 = \{f_1, f_1a\}$, $\mathfrak{S}_2 = \{f_2, f_2a\}$, $\mathfrak{S}_3 = \{f_1, f_2\}$. Hence, $\widetilde{\Gamma}(m) = \left\{f_1, f_2, f_1a, f_2a, f_1a^2, f_2a^2\right\}$, so that

$$\widehat{\Phi} : \mathrm{Rep}(L_1, 2, 2) \to \mathrm{Mat}_{6\times 2}(\Bbbk), \quad (a, f_1, f_2) \mapsto \widehat{\Phi}_{(a,f_1,f_2)} = \begin{pmatrix} f_1 \\ f_2 \\ f_1a \\ f_2a \\ f_1a^2 \\ f_2a^2 \end{pmatrix}.$$

In particular, $\widehat{J} = \Bbbk^6$.

First of all we describe the subset $\operatorname{Im}\widehat{\Phi} = X(\mathfrak{S}_1)\cup X(\mathfrak{S}_2)\cup X(\mathfrak{S}_3) \subseteq \mathrm{Mat}_{6\times 2}(\Bbbk)$. The chart $X(\mathfrak{S}_1)$ consists of 6×2-matrices $B=(b_{ij})$ with $|B(\mathfrak{S}_1)| = \left|\begin{smallmatrix} b_{11} & b_{12} \\ b_{31} & b_{32}\end{smallmatrix}\right| \neq 0$ satisfying conditions $B_{f_2a} = B_{f_2}B(\mathfrak{S}_1)^{-1}B(\mathfrak{S}_1 a)$ and $B_{f_2a^2} = B_{f_2a}B(\mathfrak{S}_1)^{-1}B(\mathfrak{S}_1 a)$, that may be expanded as

$$(b_{41}, b_{42}) = (b_{21}, b_{22})\begin{pmatrix} b_{11} & b_{12} \\ b_{31} & b_{32}\end{pmatrix}^{-1}\begin{pmatrix} b_{31} & b_{32} \\ b_{51} & b_{52}\end{pmatrix}$$

and

$$(b_{61}, b_{62}) = (b_{41}, b_{42})\begin{pmatrix} b_{11} & b_{12} \\ b_{31} & b_{32}\end{pmatrix}^{-1}\begin{pmatrix} b_{31} & b_{32} \\ b_{51} & b_{52}\end{pmatrix}$$

respectively. Similarly we find that $X(\mathfrak{S}_2) \subseteq \mathrm{Mat}_{6\times 2(\Bbbk)}$ consists of matrices $B=(b_{ij})$ with $|B(\mathfrak{S}_2)| = \left|\begin{smallmatrix} b_{21} & b_{22} \\ b_{41} & b_{42}\end{smallmatrix}\right| \neq 0$ and

$$\begin{pmatrix} b_{31} & b_{32} \\ b_{51} & b_{52}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} b_{21} & b_{22} \\ b_{41} & b_{42}\end{pmatrix}^{-1}\begin{pmatrix} b_{41} & b_{42} \\ b_{61} & b_{62}\end{pmatrix}.$$

Finally, $X(\mathfrak{S}_3) \subseteq \mathrm{Mat}_{6\times 2}(\Bbbk)$ is a subset consisting of matrices $B=(b_{ij})$ satisfying $B(\mathfrak{S}_3) = \left|\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{smallmatrix}\right| \neq 0$ and

$$\begin{pmatrix} b_{51} & b_{52} \\ b_{61} & b_{62}\end{pmatrix} = \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42}\end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}^{-1}\begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42}\end{pmatrix}.$$

Next, we write down the quotient maps $\pi_{\mathfrak{S}_i} : \mathrm{Rep}(L_1, \mathfrak{S}_i) \cong X(\mathfrak{S}_i) \to \mathrm{Mat}_{2\times 2}(\Bbbk)$;

$$\pi_{\mathfrak{S}_1} : (a, f_1, f_2) \mapsto \begin{pmatrix} f_2 \\ f_1a^2\end{pmatrix}\begin{pmatrix} f_1 \\ f_1a\end{pmatrix}^{-1},\ B=(b_{ij}) \mapsto \begin{pmatrix} b_{31} & b_{32} \\ b_{51} & b_{52}\end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{31} & b_{32}\end{pmatrix}^{-1};$$

$$\pi_{\mathfrak{S}_2} : (a, f_1, f_2) \mapsto \begin{pmatrix} f_1 \\ f_2a^2\end{pmatrix}\begin{pmatrix} f_2 \\ f_2a\end{pmatrix}^{-1},\ B=(b_{ij}) \mapsto \begin{pmatrix} b_{41} & b_{42} \\ b_{61} & b_{62}\end{pmatrix}\begin{pmatrix} b_{21} & b_{22} \\ b_{41} & b_{42}\end{pmatrix}^{-1};$$

$$\pi_{\mathfrak{S}_3} : (a, f_1, f_2) \mapsto \begin{pmatrix} f_1a \\ f_2a\end{pmatrix}\begin{pmatrix} f_1 \\ f_2\end{pmatrix}^{-1},\ B=(b_{ij}) \mapsto \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42}\end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}^{-1}.$$

Each $\pi_{\mathfrak{S}_i}$ admits a section $\mathrm{Mat}_{2\times 2}(\Bbbk) \to \mathrm{Rep}(L_1, \mathfrak{S}_i)$ sending $C = (x^{(i)}_{ab})$ to the triple (a^C, f_1^C, f_2^C) with

$$a^C = \begin{pmatrix} 0 & 1 \\ x^{(1)}_{21} & x^{(1)}_{22}\end{pmatrix},\ f_1^C = (1,0),\ f_2^C = (x^{(1)}_{11}, x^{(1)}_{12}),\ \text{for } i=1 \tag{4.1}$$

$$a^C = \begin{pmatrix} 0 & 1 \\ x^{(2)}_{21} & x^{(2)}_{22}\end{pmatrix},\ f_1^C = (x^{(2)}_{11}, x^{(2)}_{12}),\ f_2^C = (1,0),\ \text{for } i=2 \tag{4.2}$$

$$a^C = C = \begin{pmatrix} x^{(1)}_{11} & x^{(1)}_{12} \\ x^{(1)}_{21} & x^{(1)}_{22}\end{pmatrix},\ f_1^C = (1,0),\ f_2^C = (0,1),\ \text{for } i=3. \tag{4.3}$$

Now, if a triple $T := (a, f_1, f_2)$ belongs to $\mathrm{Rep}(Q, \mathfrak{S}_i)\cap\mathrm{Rep}(Q, \mathfrak{S}_j)$, for some $i\neq j$, we obtain two normal forms $(a^{\pi_{\mathfrak{S}_i}(T)}, f_1^{\pi_{\mathfrak{S}_i}(T)}, f_2^{\pi_{\mathfrak{S}_i}(T)})$ and $(a^{\pi_{\mathfrak{S}_j}(T)}, f_1^{\pi_{\mathfrak{S}_j}(T)}, f_2^{\pi_{\mathfrak{S}_j}(T)})$ corresponding to 2×2-matrices $\pi_{\mathfrak{S}_i}(T) = (x^{(i)}_{ab})$ and $\pi_{\mathfrak{S}_j}(T) = (x^{(j)}_{ab})$. One easily computes that

$$\begin{pmatrix} x^{(1)}_{11} & x^{(1)}_{12} \\ x^{(1)}_{21} & x^{(1)}_{22}\end{pmatrix} = \begin{pmatrix} f_2 \\ f_1a^2\end{pmatrix}\begin{pmatrix} f_1 \\ f_1a\end{pmatrix}^{-1} = \begin{pmatrix} -\dfrac{x^{(3)}_{11}}{x^{(3)}_{12}} & \dfrac{1}{x^{(3)}_{12}} \\ x^{(3)}_{12}x^{(3)}_{21} - x^{(3)}_{11}x^{(3)}_{22} & x^{(3)}_{11} + x^{(3)}_{22}\end{pmatrix},$$

$$\begin{pmatrix} x_{11}^{(3)} & x_{12}^{(3)} \\ x_{21}^{(3)} & x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} -\dfrac{x_{11}^{(1)}}{x_{12}^{(1)}} & \dfrac{1}{x_{12}^{(1)}} \\ \dfrac{(x_{12}^{(1)})^2 x_{21}^{(1)} - (x_{11}^{(1)})^2 - x_{11}^{(1)} x_{12}^{(1)} x_{22}^{(1)}}{x_{12}^{(1)}} & \dfrac{x_{11}^{(1)} + x_{12}^{(1)} x_{22}^{(1)}}{x_{12}^{(1)}} \end{pmatrix},$$

$$\begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} -\dfrac{x_{11}^{(1)} + x_{12}^{(1)} x_{22}^{(1)}}{(x_{12}^{(1)})^2 x_{21}^{(1)} - (x_{11}^{(1)})^2 - x_{11}^{(1)} x_{12}^{(1)} x_{22}^{(1)}} & \dfrac{x_{12}^{(1)}}{(x_{12}^{(1)})^2 x_{21}^{(1)} - (x_{11}^{(1)})^2 - x_{11}^{(1)} x_{12}^{(1)} x_{22}^{(1)}} \\ x_{21}^{(1)} & x_{22}^{(1)} \end{pmatrix}.$$

We return for a while to the case of arbitrary m, q and k. Since the quotient is embedded into $\mathrm{Gr}_m(\widehat{J})$, we need to introduce a connection between local coordinates on $Y(\mathfrak{S}) := X(\mathfrak{S}) /\!\!/ GL(\alpha)$ and Plücker coordinates on $\mathrm{Gr}_m(\widehat{J})$. Observe that the latter are of the form $p_{\mathfrak{R}}$, for all subsets $\mathfrak{R} \subseteq \widetilde{\Gamma}(m)$ of cardinality m. Indeed, the natural projection $\mathrm{Mat}_{k(m+1)^q \times m}(\Bbbk) \supseteq \mathrm{IHom}_m(\widehat{J}) \twoheadrightarrow \mathrm{Gr}_m(\widehat{J})$ maps a matrix B to a point ω_B, whose Plücker coordinates are $m \times m$-minors of B. So, we denote by $p_{\mathfrak{R}}$ the coordinate with corresponding minor consisting of rows prescribed by $\mathfrak{R}$. As for the local coordinates on $Y(\mathfrak{S})$, their definition together with Cramer's rule imply that they are indexed by m-element subsets $\mathfrak{R}$ in $\widetilde{\Gamma}(m)$ that may be obtained from $\mathfrak{S}$ by replacement of one of its elements by an element of $(\mathfrak{S}a \backslash \mathfrak{S}) \cup \{f_i \notin \mathfrak{S}\}$.

In Example 4.2, we have $\mathrm{Gr}_m(\widehat{J}) = \mathrm{Gr}_2(\Bbbk^6)$. As it was suggested before, we index Plücker coordinates by pairs of paths. For instance, $p_{f_2, f_1 a^2}$ stands for p_{25}. For a point ω_B corresponding to a rank m matrix $B \in \mathrm{Mat}_{6\times 2}(\Bbbk)$ it comes as $\begin{vmatrix} b_{21} & b_{22} \\ b_{51} & b_{52} \end{vmatrix}$, and this equals $\begin{vmatrix} f_2 \\ f_1 a^2 \end{vmatrix}$ if $(b_{ij}) = \widehat{\Phi}((a, f_1, f_2))$. We obtain the expressions

$$\begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} \dfrac{p_{f_2,f_1a}}{p_{f_1,f_1a}} & -\dfrac{p_{f_1,f_2}}{p_{f_1,f_1a}} \\ -\dfrac{p_{f_1a,f_1a^2}}{p_{f_1,f_1a}} & \dfrac{p_{f_1,f_1a^2}}{p_{f_1,f_1a}} \end{pmatrix}, \quad \begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} \dfrac{p_{f_1,f_2a}}{p_{f_2,f_2a}} & -\dfrac{p_{f_1,f_2}}{p_{f_2,f_2a}} \\ -\dfrac{p_{f_2a,f_2a^2}}{p_{f_2,f_2a}} & \dfrac{p_{f_2,f_2a^2}}{p_{f_2,f_2a}} \end{pmatrix},$$

$$\begin{pmatrix} x_{11}^{(3)} & x_{12}^{(3)} \\ x_{21}^{(3)} & x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} -\dfrac{p_{f_2,f_1a}}{p_{f_1,f_2}} & \dfrac{p_{f_1,f_1a}}{p_{f_1,f_2}} \\ -\dfrac{p_{f_2,f_2a}}{p_{f_1,f_2}} & \dfrac{p_{f_1,f_2a}}{p_{f_1,f_2}} \end{pmatrix}.$$

Now, we determine the equations in Plücker coordinates that define the closure of the quotient in $\mathrm{Gr}_2(\widehat{J})$ in Example 4.2. First of all, there are Plücker relations. We also have the following equations coming from transition relations between $x_{ab}^{(i)}$ and $x_{cd}^{(j)}$

$$p_{f_1a,f_1a^2} p_{f_2,f_2a} = p_{f_2a,f_2a^2} p_{f_1,f_1a}, \quad p_{f_1,f_1a^2} p_{f_2,f_2a} = p_{f_2,f_2a^2} p_{f_1,f_1a},$$
$$p_{f_1a,f_1a^2} p_{f_1,f_2}^2 = p_{f_1,f_1a}(p_{f_1,f_1a} p_{f_2,f_2a} - p_{f_2,f_1a} p_{f_1,f_2a}),$$
$$p_{f_2a,f_2a^2} p_{f_1,f_2}^2 = p_{f_2,f_2a}(p_{f_1,f_1a} p_{f_2,f_2a} - p_{f_2,f_1a} p_{f_1,f_2a}),$$
$$p_{f_1,f_1a^2} p_{f_1,f_2} = p_{f_1,f_1a}(p_{f_1,f_2a} - p_{f_2,f_1a}),$$
$$p_{f_2,f_2a^2} p_{f_1,f_2} = p_{f_2,f_2a}(p_{f_1,f_2a} - p_{f_2,f_1a}).$$

As we see, not all $p_{\mathfrak{R}}$, $\mathfrak{R} \subseteq \widetilde{\Gamma}(m)$, occur as numerators of local coordinates on the quotient. Those that do not occur we will call *exceed* and others *essential*. We claim that exceed homogeneous coordinates may be eliminated, i.e., that we are able to express them as polynomials in local coordinates in each affine chart. Indeed, we may express f_i and a_j, and then compute all maximal minors of the matrix of $\Phi_{(a,f)}$. For instance, we have

$$\frac{p_{f_1a,f_2a}}{p_{f_1,f_1a}} = \frac{p_{f_1,f_2} p_{f_1a,f_1a^2}}{p_{f_1,f_1a^2}^2}, \qquad \frac{p_{f_1a,f_2a}}{p_{f_2,f_2a}} = \frac{p_{f_1,f_2} p_{f_2a,f_2a^2}}{p_{f_2,f_2a^2}^2},$$

$$\frac{p_{f_1a,f_2a}}{p_{f_1,f_2}} = \frac{p_{f_1,f_1a}p_{f_2,f_2a} - p_{f_1,f_2a}p_{f_1a,f_2}}{p^2_{f_1,f_2}},$$

$$\frac{p_{f_1,f_2a^2}}{p_{f_1,f_1a}} = \frac{p_{f_1,f_2}p_{f_1a,f_1a^2}p_{f_1,f_1a} + p_{f_2,f_1a}p_{f_1,f_1a^2}p_{f_1,f_1a} - p_{f_1,f_2}p^2_{f_1,f_1a^2}}{p^3_{f_1,f_1a}},$$

$$\frac{p_{f_1,f_2a^2}}{p_{f_1,f_2}} = \frac{p^2_{f_1,f_2a} - p_{f_1,f_1a}p_{f_2,f_2a}}{p^2_{f_1,f_2}}.$$

Returning back to the general case, we may consider an obvious projection-like map $\mathcal{M}^s(L_q,m,k) \to \mathbb{P}^N$, where $N+1$ is the number of non-exceed coordinates. Clearly it is well defined. Denote by $\widehat{Y}_0$ the image of the quotient in $\mathbb{P}^N$. We claim now that the closure $\widehat{Y}$ of $\widehat{Y}_0$ (or at least something containing $\widehat{Y}$ as an irreducible component) is defined in $\mathbb{P}^N$ solely by the equations that come from transition relations on the quotient, i.e., that Plücker equations are no more required. The reason is that on each affine chart $Y(\mathfrak{S})$, where $\mathfrak{S}$ is a J-skeleton, we can express all non-exceed coordinates $p_{\mathfrak{R}}$ in local coordinates of $Y(\mathfrak{S})$ using only this kind of equations, and since $Y(\mathfrak{S})$ is an affine space, these local coordinates are algebraically independent. So, no additional equations are needed.

In the case $q=1$, $m=k=2$ we have 15 Plücker coordinates, only 9 of them are essential and other 6 are exceed. So, the quotient may be embedded as a locally closed subset into $\mathbb{P}^8$.

Example 4.3. Let $q=m=2$, $k=1$. Denote the loops by a and b. There are then two possible J-skeleta: $\mathfrak{S}_1 = \{f, fa\}$ and $\mathfrak{S}_2 = \{f, fb\}$. Hence $\widetilde{\Gamma}(m) = \{f, fa, fa^2, fb, fab, fba, fb^2\}$ (we fix this order of paths and construct the map $\widehat{\Phi}$ according to it) and $\widehat{J} = \Bbbk^7$.

It is not hard to see that

$$X(\mathfrak{S}_1) = \left\{ B \in \mathrm{Mat}_{7\times 2}(\Bbbk) \;\middle|\; \begin{array}{l} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \neq 0, \\ \begin{pmatrix} b_{61} & b_{62} \end{pmatrix} = \begin{pmatrix} b_{31} & b_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1} \begin{pmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \end{pmatrix} \\ \begin{pmatrix} b_{71} & b_{72} \end{pmatrix} = \begin{pmatrix} b_{31} & b_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1} \begin{pmatrix} b_{31} & b_{32} \\ b_{51} & b_{52} \end{pmatrix} \end{array} \right\}$$

and

$$X(\mathfrak{S}_2) = \left\{ B \in \mathrm{Mat}_{7\times 2}(\Bbbk) \;\middle|\; \begin{array}{l} \begin{vmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{vmatrix} \neq 0, \\ \begin{pmatrix} b_{41} & b_{42} \end{pmatrix} = \begin{pmatrix} b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{pmatrix}^{-1} \begin{pmatrix} b_{21} & b_{22} \\ b_{61} & b_{62} \end{pmatrix} \\ \begin{pmatrix} b_{51} & b_{52} \end{pmatrix} = \begin{pmatrix} b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{pmatrix}^{-1} \begin{pmatrix} b_{31} & b_{32} \\ b_{71} & b_{72} \end{pmatrix} \end{array} \right\}$$

The quotient maps are

$$\pi_{\mathfrak{S}_1} : (a,b,f) \mapsto \begin{pmatrix} fa^2 \\ fb \\ fab \end{pmatrix} \begin{pmatrix} f \\ fa \end{pmatrix}^{-1}, \; B = (b_{ij}) \mapsto \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1},$$

$$\pi_{\mathfrak{S}_2} : (a,b,f) \mapsto \begin{pmatrix} fa \\ fba \\ fb^2 \end{pmatrix} \begin{pmatrix} f \\ fb \end{pmatrix}^{-1}, \; B = (b_{ij}) \mapsto \begin{pmatrix} b_{21} & b_{22} \\ b_{61} & b_{62} \\ b_{71} & b_{72} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{pmatrix}^{-1},$$

so that $X(\mathfrak{S}_1)/\!\!/\operatorname{GL}(\alpha) \cong X(\mathfrak{S}_1)/\!\!/\operatorname{GL}(\alpha) \cong \operatorname{Mat}_{3\times 2}(\Bbbk) \cong \mathbb{A}^6$. The sections $s_i : \operatorname{Mat}_{3\times 2}(\Bbbk) \to X(\mathfrak{S}_i) \xrightarrow{\sim} \operatorname{Rep}(Q, \mathfrak{S}_i)$ are as follows:

$$\mathfrak{s}_1 : \begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} \\ x_{31}^{(1)} & x_{32}^{(1)} \end{pmatrix} \mapsto \left(\begin{pmatrix} 0 & 1 \\ x_{11}^{(1)} & x_{12}^{(1)} \end{pmatrix}, \begin{pmatrix} x_{21}^{(1)} & x_{22}^{(1)} \\ x_{31}^{(1)} & x_{32}^{(1)} \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right),$$

$$\mathfrak{s}_2 : \begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \\ x_{31}^{(2)} & x_{32}^{(2)} \end{pmatrix} \mapsto \left(\begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ x_{31}^{(2)} & x_{32}^{(2)} \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right),$$

with transition functions

$$\begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} \\ x_{31}^{(1)} & x_{32}^{(1)} \end{pmatrix} = \begin{pmatrix} x_{12}^{(2)}x_{21}^{(2)} - x_{11}^{(2)}x_{22}^{(2)} & x_{11}^{(2)} + x_{22}^{(2)} \\ -\dfrac{x_{11}^{(2)}}{x_{12}^{(2)}} & \dfrac{1}{x_{12}^{(2)}} \\ \dfrac{(x_{12}^{(2)})^2 x_{31}^{(2)} - (x_{11}^{(2)})^2 - x_{11}^{(2)}x_{12}^{(2)}x_{32}^{(2)}}{x_{12}^{(2)}} & \dfrac{x_{11}^{(2)} + x_{12}^{(2)}x_{32}^{(2)}}{x_{12}^{(2)}} \end{pmatrix}$$

and

$$\begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \\ x_{31}^{(2)} & x_{32}^{(2)} \end{pmatrix} = \begin{pmatrix} -\dfrac{x_{21}^{(1)}}{x_{22}^{(2)}} & \dfrac{1}{x_{22}^{(1)}} \\ \dfrac{(x_{22}^{(1)})^2 x_{11}^{(1)} - (x_{21}^{(1)})^2 - x_{12}^{(1)}x_{22}^{(1)}x_{21}^{(1)}}{x_{22}^{(1)}} & \dfrac{x_{21}^{(1)} + x_{12}^{(1)}x_{22}^{(1)}}{x_{22}^{(1)}} \\ x_{22}^{(1)}x_{31}^{(1)} - x_{21}^{(1)}x_{32}^{(1)} & x_{21}^{(1)} + x_{32}^{(1)} \end{pmatrix}$$

Having

$$\begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} \\ x_{31}^{(1)} & x_{32}^{(1)} \end{pmatrix} = \begin{pmatrix} -\frac{p_{fa,fa^2}}{p_{f,fa}} & \frac{p_{f,fa^2}}{p_{f,fa}} \\ -\frac{p_{fa,fb}}{p_{f,fa}} & \frac{p_{f,fb}}{p_{f,fa}} \\ -\frac{p_{fa,fab}}{p_{f,fa}} & \frac{p_{f,fab}}{p_{f,fa}} \end{pmatrix}, \quad \begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \\ x_{31}^{(2)} & x_{32}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{p_{fa,fb}}{p_{f,fb}} & \frac{p_{f,fa}}{p_{f,fb}} \\ -\frac{p_{fb,fba}}{p_{f,fb}} & \frac{p_{f,fba}}{p_{f,fb}} \\ -\frac{p_{fb,fb^2}}{p_{f,fb}} & \frac{p_{f,fb^2}}{p_{f,fb}} \end{pmatrix},$$

we obtain the following set of equations

$$p_{fa,fa^2}p_{f,fb}^2 + p_{f,fa}(p_{fa,fb}p_{f,fba} + p_{f,fa}p_{fb,fba}) = 0,$$

$$p_{fb,fb^2}p_{f,fa}^2 - p_{f,fa}(p_{f,fb}p_{fa,fab} - p_{fa,fb}p_{f,fab}) = 0,$$

$$p_{f,fa^2}p_{f,fb} - p_{f,fa}(p_{fa,fb} + p_{f,fba}) = 0,$$

$$p_{f,fb^2}p_{f,fa} - p_{f,fb}(p_{f,fab} - p_{fa,fb}) = 0.$$

So, out of 21 possible coordinates only 11 are essential (i.e., occur as numerators of local coordinates on the quotient) and other 10 ones are exceed. Therefore, the quotient is a locally closed subset in $\mathbb{P}^{10}$ obtained by intersection of nonzero loci of $p_{f,fa}$ and $p_{f,fb}$ with a closed subset defined by the above equations. In particular, $\widehat{Y}$ is a complete intersection.

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